

Reduction of the planar 4-vortex system at zero momentum

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Abstract

The system of four point vortices in the plane has relative equilibria that behave as composite particles, in the case where three of the vortices have strength $-\Gamma/3$ and one of the vortices has strength Γ . These relative equilibria occur at nongeneric momenta. The reduction of this system, at those momenta, by continuous and then discrete symmetries, classifies the 4-vortex states which have been observed as products of collisions of two such composite particles. In this article I explicitly calculate these reductions, and show they are qualitatively identical one degree of freedom systems on a cylinder. The flows on these reduced systems all have one stable equilibrium and one unstable equilibrium, and all the orbits are periodic except for two homoclinic connections to the unstable equilibrium.

In the system of N vortices in the plane the n^{th} vortex has location $z_n = x_n + iy_n \in P^{\text{pl}} \equiv (\mathbb{C}^2)^N$. The Hamiltonian and symplectic form are

$$H^{\text{pl}} \equiv -\frac{1}{4\pi} \sum_{m < n} \Gamma_n \Gamma_m \ln |z_n - z_m|^2, \quad \omega^{\text{pl}} \equiv \bigoplus_{n=1}^N \Gamma_n \omega_0, \quad (1)$$

where $\omega_0(a, b) \equiv -\text{Im}(a\bar{b})$ for complex numbers $a, b \in \mathbb{C}$. This system admits the symmetry group $SE(2) = \{(e^{i\theta}, a)\}$ of Euclidean symmetries acting diagonally on each factor \mathbb{C} of P^{pl} by $(e^{i\theta}, a) \cdot z \equiv e^{i\theta} z + a$. A momentum mapping is

$$J^{\text{pl}} \equiv -\sum_{n=1}^N \Gamma_n \begin{bmatrix} \frac{1}{2}|z_n|^2 \\ iz_n \end{bmatrix}, \quad (2)$$

where $\mathfrak{se}(2)^*$ is identified with $\mathfrak{se}(2) = \{(\dot{\theta}, \dot{a})\} \cong \mathbb{R}^3$ by the standard inner product of \mathbb{R}^3 . For more details on the system of N vortices on the sphere or

plane, see Kidambi and Newton [1998], Pekarsky and Marsden [1998], and the references therein.

In Patrick [2000] I have shown that the point of phase space corresponding to a central vortex of strength Γ at the origin surrounded at distance α by three symmetrically placed outer vortices of strength $-\Gamma/3$ is a formally stable relative equilibrium for this system; that is

$$z_1 = \alpha, \quad z_2 = \alpha e^{2\pi i/3}, \quad z_3 = \alpha e^{-2\pi i/3}, \quad z_4 = 0 \quad (3)$$

is a formally stable relative equilibrium of the planar point vortex system with $N = 4$ and

$$\Gamma_1 = \Gamma_2 = \Gamma_3 = -\frac{\Gamma}{3}, \quad \Gamma_4 = \Gamma.$$

The momentum mapping is equivariant for these vortex strengths because the sum of the vortex strengths is zero. The relative equilibrium itself does not translate but merely rotates with generator $\dot{\theta}_e \equiv \Gamma/3\pi\alpha^2$, and by direct substitution the momentum is $\mu_e \equiv (\Gamma\alpha^2/2, 0)$. This momentum value is nongeneric, since it has isotropy group all of $SE(2)$, and the position of this relative equilibrium is unstable under small perturbation to nonzero momentum: as discussed in Patrick [2000], the relative equilibria will move about on the plane as a composite particle. To date these are the only relative equilibria of point vortices which are both formally stable and have nongeneric momentum. The particular system of four vortices with the strengths Γ_i is singled out by this fact.

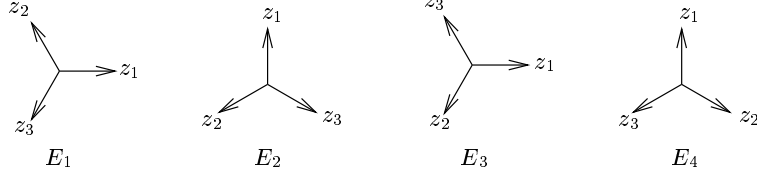
When two such relative equilibria collide the result may be localized 4-vortex states far from such relative equilibria but each near to momentum μ_e , and these too have been observed to move about on the plane as composite particles. The phase space of classifying these states, since they are far from the relative equilibrium (3), yet still close to momentum μ_e , will be the reduced phase space of the 4-vortex system at momentum μ_e . This phase space has dimension $4 \times 2 - 6 = 2$. Here I calculate this phase space and, qualitatively, its associated Hamiltonian flow. One can find other reductions on various point vortex systems, in the sphere and in the plane, in Adams and Ratiu [1988], Aref and Pomphrey [1982], Eckhardt and Aref [1988], and Pekarsky and Marsden [1998]. The general issue of drifting relative equilibria at nongeneric momenta is discussed at length in the sequence of articles Patrick [1992], Patrick [1996], Patrick [1995], and Patrick [1999].

To calculate the quotient

$$\pi_{\mu_e} : (J^{\text{Pl}})^{-1}(\mu_e) \rightarrow (P^{\text{Pl}})_{\mu_e} \equiv (J^{\text{Pl}})^{-1}(\mu_e)/SE(2),$$

one may first translate the central vortex to the origin, whereupon, by (2), the sum of the outer vortex positions must vanish. The resulting 4-dimensional vector space $\mathbb{C}^2 = \{(u_1 + iv_1, u_2 + iv_2)\}$ is spanned by the following orthonormal

basis $\{E_i\}$ of outer vortex positions:



or more specifically the column vectors of the matrix

$$E \equiv \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

The $SE(2)$ symmetry then remaining on \mathbb{C}^2 is $SO(2)$ acting by diagonal clockwise rotation, and since the above basis is orthonormal, the level set of J^{pl} becomes

$$\{(u_1 + iv_1, u_2 + iv_2) : u_1^2 + v_1^2 + u_2^2 + v_2^2 = 3\alpha^2\}.$$

The Hopf variables

$$\begin{aligned} w_1 &\equiv \frac{2}{3\alpha^2}(u_1v_2 - u_2v_1), & w_3 &\equiv \frac{-1}{3\alpha^2}((u_1^2 + v_1^2) - (u_2^2 + v_2^2)), \\ w_2 &\equiv \frac{2}{3\alpha^2}(u_1u_2 + v_1v_2), & w_4 &\equiv \frac{1}{3\alpha^2}((u_1^2 + v_1^2) + (u_2^2 + v_2^2)), \end{aligned}$$

with image the semi-algebraic set

$$w_1^2 + w_2^2 + w_3^2 - w_4^2 = 0, \quad w_4 \geq 0.$$

are a quotient map for the diagonal $SO(2)$ action on this space, so restricting these to the $w_4 = 1$ sphere gives the quotient as

$$(P^{\text{pl}})_{\mu_e} = \{(w_1, w_2, w_3) : w_1^2 + w_2^2 + w_3^2 = 1\}.$$

One partial section to the Hopf mapping, obtained by imposing $v_1 = 0$, is given by

$$u_1 = \frac{\sqrt{3}\alpha}{2}\sqrt{1-w_3}, \quad v_1 = 0, \quad u_2 = \frac{\sqrt{3}\alpha}{2}\frac{w_2}{\sqrt{1-w_3}}, \quad v_2 = \frac{\sqrt{3}\alpha}{2}\frac{w_1}{\sqrt{1-w_3}},$$

and a section to the map from P^{pl} to \mathbb{C}^2 may be obtained by setting $z_4 = 0$ and then multiplying the vector (u_1, v_1, u_2, v_2) by E to obtain the triple (z_1, z_2, z_3) . Using this section to pull back the symplectic form on P^{pl} gives the reduced symplectic form

$$\omega_{\mu_e} = \frac{\Gamma\alpha^2}{4}\omega_{S^2}, \quad \omega_{S^2}(w)(\dot{w}_1, \dot{w}_2) \equiv -w \cdot (\dot{w}_1 \times \dot{w}_2).$$

To summarize, the symplectic reduced phase space is the 2-sphere with symplectic form the constant multiple $\Gamma\alpha^2/4$ of the standard symplectic form on the 2-sphere obtained by left-hand reduction of the cotangent bundle of $SO(3)$.

The action of the discrete group \mathfrak{S}_3 corresponding to permutation of the identical outer vortices descends to the reduced space since it commutes with the action of $SE(2)$. Using the above section this reduced discrete action is easily calculated to be the restriction to the 2-sphere of the linear representation $\sigma : \mathfrak{S}_3 \rightarrow SO(3)$ given by

$$\left\{ \begin{array}{l} \sigma_{(12)} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \sigma_{(13)} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \sigma_{(23)} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \\ \sigma_{(123)} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \sigma_{(132)} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{array} \right.$$

The squares of the interparticle distances can be descended to the reduced space, giving $\alpha^2/2$ times each of the 6 linear functionals

$$l_{12} \equiv -3(\sqrt{3}w_1 - w_2 - 2), \quad l_{13} \equiv 3(\sqrt{3}w_1 + w_2 + 2), \quad l_{23} \equiv -6(w_2 - 1) \\ l_{41} \equiv 2(w_2 + 1), \quad l_{42} \equiv -(\sqrt{3}w_1 + w_2 - 2), \quad l_{43} \equiv \sqrt{3}w_1 - w_2 + 2$$

and in terms of these the reduced Hamiltonian is

$$H_{\mu_e} = \frac{\Gamma^2}{36\pi} \ln \left(\left(\frac{\alpha^2}{2} \right)^6 \frac{(l_{41}l_{42}l_{43})^3}{l_{12}l_{13}l_{23}} \right).$$

The linear functionals l_{ij} are independent of w_3 and are zero for the two equilateral triangles circumscribing the unit circle in the (w_1, w_2) plane (see the left of Figure (1)). The Hamiltonian is finite except for the points in this plane where the triangles touch the circle, where it is positive infinity on the triangle that points downward, corresponding to collision states of two outer vortices, and negative infinity on the triangle pointing upward, corresponding to collisions states of an outer vortex with the central vortex. The Hamiltonian has an amusing relationship with the geometry of the collision triangles: it differs by a constant from the function that is the logarithm of the product of the three distances to the $-\infty$ -collision triangle, cubed, divided by the product of the three distances to the $+\infty$ -collision triangle.

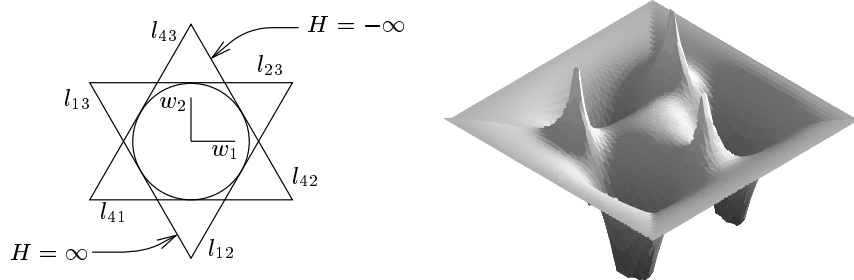


Figure 1: Left: the level sets of the linear functionals l_{ij} are the triangles shown. Right: The reduced Hamiltonian H_{μ_e} ; the sphere has been stereographically projected from the south pole and then the plane compressed to a square, so that the boundary of the square is the south pole. Clearly visible are the $\pm\infty$ collision states and at the center the stable equilibrium. Another stable equilibrium was, on the original sphere, located at the south pole, so it occupies the boundary of the square. Between the $+\infty$ collision “mountains” and the stable equilibrium at the center are three saddle points and there are three other saddle points between the $+\infty$ collision “mountains” and the edge of the square.

The equilibria, and hence the relative equilibria of the unreduced system, may be found by calculating the critical points of H_{μ_e} . The south pole is the stable equilibrium corresponding to the formally stable relative equilibrium (3), and the north pole corresponds to the mirror image of this, obtained by exchanging z_2 and z_3 in (3). There are six unstable equilibria, namely the points of the orbit of the equilibrium $w_1 = 0$, $w_2 = \sqrt{3} - 1$ under the action of \mathfrak{S}_3 . One relative equilibrium corresponding to these unstable equilibria is

$$z_1 = \alpha \sqrt[4]{3}, \quad z_2 = -\frac{\alpha}{2} \sqrt[4]{3} + \frac{\alpha}{2\sqrt{2}}(\sqrt{3} - 3)i, \quad z_3 = \bar{z}_2, \quad z_4 = 0. \quad (4)$$

A graph of the reduced Hamiltonian as a function of w_1 and w_2 is shown on the right of Figure (1).

The above is a complete analysis of the reduced space and the phase portrait. However, to find the most succinct classification of the localized 4-vortex states, one should also quotient by the action of \mathfrak{S}_3 . Then the unstable equilibria will collapse to a single point, the two stable equilibria collapse to a single point, and there will be one $+\infty$ collision state and one $-\infty$ collision state. It is easy to calculate that the stable equilibria have isotropy the cyclic group \mathfrak{Z}_3 , the collisions have isotropy the cyclic group \mathfrak{Z}_2 , and these are the only points of the reduced phase space with isotropy.

To find the quotient of the reduced space by \mathfrak{S}_3 one can find its image under sufficiently many functions invariant under the action of \mathfrak{S}_3 . By averaging up to degree 4 polynomials in w_1, w_2, w_3 , one finds the invariant polynomials

$$\begin{aligned} p_1 &\equiv w_3^2, & p_2 &\equiv w_2(3w_1^2 - w_2^2), \\ p_3 &\equiv w_1w_3(3w_2^2 - w_1^2), & p_4 &\equiv w_1^2 + w_2^2 + w_3^2 \end{aligned}$$

and the relations

$$p_1((p_1 - p_4)^3 + p_2^2) + p_3^2 = 0, \quad 0 \leq p_1 \leq p_4. \quad (5)$$

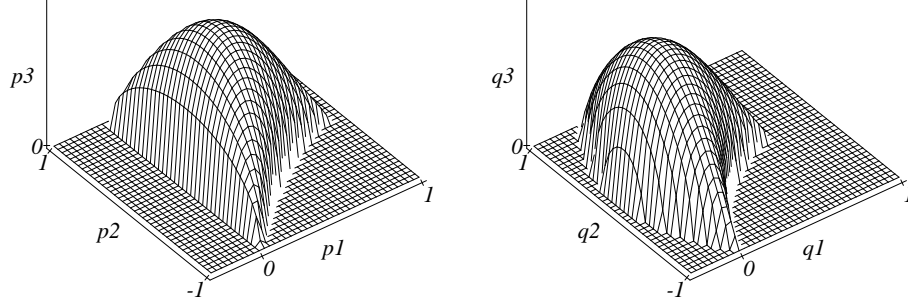


Figure 2: Left: top half of the semi-algebraic set (6). The $+\infty$ collision state is to the front and the $-\infty$ collision state is behind and obscured; the singularity at $(p_1, p_2, p_3) = (1, 0, 0)$ is the relative equilibrium (3). Right: same as left but deformed according to (7). The resulting surface may be radially projected to the sphere and then stereographically projected to the plane from the point $(q_1, q_2, q_3) = (0, -1, 0)$.

The map $(w_1, w_2, w_3) \mapsto (p_1, p_2, p_3, p_4)$ can be seen as the composition of simpler steps, namely

$$\tilde{p}_1 \equiv w_3, \quad \tilde{p}_2 + i\tilde{p}_3 \equiv (-w_2 + iw_1)^3, \quad \tilde{p}_4 \equiv w_1^2 + w_2^2 + w_3^2,$$

followed by

$$p_1 = \tilde{p}_1^2, \quad p_2 = \tilde{p}_2, \quad p_3 = \tilde{p}_1\tilde{p}_3, \quad p_4 = \tilde{p}_4,$$

and hence it is easily seen that $(w_1, w_2, w_3) \mapsto (p_1, p_2, p_3, p_4)$, with image the semi-algebraic set (5), is a quotient map for the action of \mathfrak{S}_3 on \mathbb{C}^2 . Thus (5) restricts to a quotient map of the reduced space to the semi-algebraic set

$$p_1((p_1 - 1)^3 + p_2^2) + p_3^2 = 0, \quad 0 \leq p_1 \leq 1. \quad (6)$$

The Hamiltonian this reduced space is

$$H_{\mu_e} = \frac{\Gamma^2}{36\pi} \ln \left(\frac{\alpha^{12}}{2^4 3^3} \frac{(1 + 3p_1 - p_2)^3}{1 + 3p_1 + p_2} \right).$$

The set (6), the top half of which is graphed on the left of Figure (2), is homeomorphic to a two sphere and has three singularities, corresponding to the two collision types of the reduced phase space and to the stable relative equilibrium (3). If the $\pm\infty$ -collision states are removed then the phase space is a cylinder; however, the symplectic form of this phase space is undefined at one point, namely the stable equilibrium corresponding to the relative equilibrium (3). An explicit map to this cylinder can be constructed by deforming (6) according to

$$q_1 \equiv p_1 - \frac{1}{4}(1 - p_2^2), \quad q_2 \equiv p_2, \quad q_3 \equiv p_3, \quad (7)$$

resulting in the surface plotted on the right of Figure (2). The collision states

$$(p_1, p_2, p_3) = (0, \pm 1, 0)$$

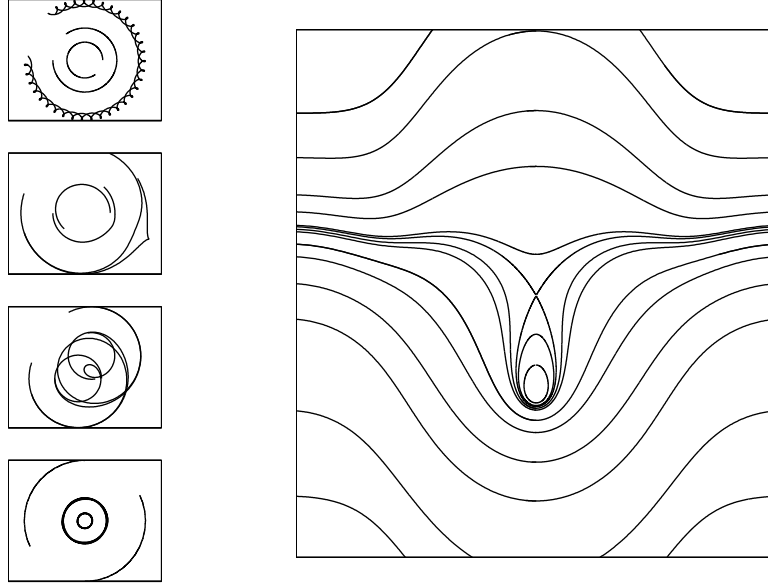


Figure 3: Right: the result of projecting, by (8), simulations of the 4-vortex system to the toral reduced space as described. The horizontal variable is θ_{μ_e} and the two vertical sides of the phase portrait should be identified. Left top to bottom: motions of the vortices on the plane near the $+\infty$ collision, the top half of the homoclinic orbit, the bottom half of the homoclinic orbit, and the $-\infty$ collision state. The pictures on the left are reconstructions of orbits on the right. Left, the third graph from the top: the motion of the vortices is a close pair for strengths -1 orbiting two vortices, one of strength 3 and the other of strength -1 , and periodically the vortex on the inside is exchanged with one of the vortices on the outside; this corresponds to the lower half of the homoclinic orbit on the right. Left, second graph from the top: the same configuration but the two outer vortices making up the close pair exchange positions; this corresponds to the upper half of the homoclinic orbit on the right.

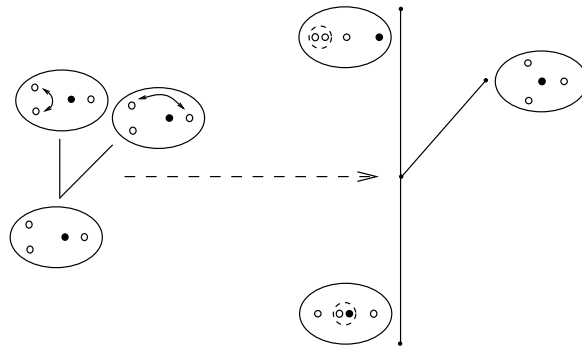


Figure 4: The space of localized 4-vortex states is the “Y” shaped union of the two lines above. The stable relative equilibrium (3) is represented by the endpoint of the line branching upwards towards the right. At the top and bottom endpoints of the vertical branch are the $\pm\infty$ collision states respectively. The bifurcation point in the middle is the unstable relative equilibrium (4). From that point proceeding upwards immediately yields to modes of motion corresponding to the two connecting homoclinic orbits, as indicated on the left.

are unchanged but the result of the deformation uniquely intersects radial half lines from the origin, and thus can be radially homeomorphed to the sphere. Then, stereographically projecting from the collision state $(q_1, q_2, q_3) = (0, -1, 0)$ results in the plane $\mathbb{C} \setminus \{0\}$, and the pairing of the angle θ_{μ_e} at the origin together with the logarithm h_{μ_e} of the radius results in a map to the 2-torus such that the $\pm\infty$ energy collision states are mapped to $\pm\infty$ respectively. The final map to the cylindrical phase space $\{(h, \theta)\}$ is then

$$h_{\mu_e} \equiv \frac{1}{2} \ln \left(\frac{q_1^2 + q_3^2}{q_2 + \sqrt{q_1^2 + q_2^2 + q_3^2}} \right), \quad \theta_{\mu_e} \equiv \arctan \left(\frac{q_3}{q_1} \right), \quad (8)$$

and Figure (3) shows the resulting phase portrait.

As stated at the beginning, the prime motivation of this article has been the classification of 4-vortex states, produced through collisions of two stable relative equilibria (3), with momenta near $\mu_e \equiv (\Gamma\alpha^2/2, 0)$ but far from the relative equilibria (3). Such localized states ought to be roughly regarded as orbits of the phase space in Figure (3), since there will be a fast time scale roughly corresponding to motion along those orbits and the localized states will present external properties corresponding to averages over that fast time scale. Thus, the space of localized 4-vortex states corresponds to the quotient space of the torus by the orbit equivalence relation of the phase space of Figure (3). This quotient is shown in Figure (4).

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